Abstract
We investigate two approaches to building sparse, adaptive representations of quantities of interest depending on uncertain parameters and deterministic design variables. We explore a nested approach, wherein we perform adaptive pseudo-spectral projections (aPSP) in the space of design variables and conduct, independently at a design point, the adaptation in the space of uncertain variables. We also develop an alternative approach, in which aPSP is conducted in the design-random parameter product space, and introduce a decomposition methodology to guide the refinement of sparse grids. Specifically, we use the decomposed projection surplus estimates to tune both the grid adaptation and selective termination criteria. We compare the performance for the two approaches in a simple test problem, and then examine their performance for a shock-tube design experiment involving a high-dimensional system of stiff ODEs. Computed results indicate that, whereas both methods provide effective means for tuning the quality of the representation in the deterministic and stochastic spaces, adaptive refinement in the product space is generally more efficient than the nested approach.

Approach 1: Nested Hierarchical Adaptation
Nested Adaptation performs local “inner-space” (u) adaptation at each realization of the “outer-space” (p) adaptation on F(p, u).
• Allows for independent tolerances in both p and u spaces
  • Good control of p space realizations
  • Ensures full tensorization of design variable and uncertain parameter spaces
  • Incurs a non-trivial cost
  • Can be easily extended to hybrid approaches combining different methods for inner or outer representations

Approach 2: Sensitivity Tuned Adaptation in Product Space
Consider product space of ξ = (p, u) and perform aPSP in that product space.
• Exploits sparsity between the two spaces
• Desirable to be able to asses and tune adaptation along different directions

Tuning adaptation:
Apply a decomposition to A where A^p is indices only along the p axis and A^u includes mixed indices.
Define:
• η^p and η^u to be a combination of η1 and η2
Adapt until either of the following is met
1: η^p ≤ Tolu and η^u ≤ Tolu
2: η^p ≤ Tol1
If the η^p or η^u fall below the respective tolerances, halt adaptation in one of two ways:
T1: Do not allow further refinement beyond the highest level reached in the converged direction but allow for additional mixed terms.
T2: Do not admit any forward neighbors in the converged direction, and consequently restrict the inclusion of many mixed terms.

Monte Carlo Error Estimate
For model F and surrogate (PCE) Ő, set χ(ξ) = F(ξ) − Ő(ξ) and define a posteriori error ζ̂ = χ̂ + χ̃ and for each space
(ζ̂)^2 = E E (χ̂)^2 = V (χ̂)^2 + E (χ̂)^2 E (F̂)^2
(ζ̃)^2 = V (χ̃)^2 + E (χ̃)^2 E (F̂)^2
where S^2 and T^2 are the first and total sensitivity indices of χ, E and V denote the expectation and variance, respectively.
Let ζ be a combination of ζ̂ and ζ̃. Estimate quantities with Monte-Carlo sampling.

Low-Dimensional Test Problem
Consider a simple test problem with a single design variable and stochastic parameter:
F(p, u) = 1 + (1/3)(1 + u/(p + 1))
× exp(−(u/(p + 1))(p + 1))
Final adaptive path and log(ζ)[u]| I for the different methods:
a posteriori ζ values:

Large Scale Demonstration: Methane Combustion
Consider a shock-tube methane combustion experiment with 22-stochastic parameters and 3 design variables. Measure peak electron concentration
Termination indicator, η and direction of η^p and η^u for the product-space adaptation:
a posteriori ζ values:

Conclusions
• Nested adaptation provides good error control but the tensor-product nature of p-u coupling makes it expensive
• Tuned product-space adaptation is a simple modification to aPSP providing enhanced adaptivity and termination criteria
• Prevents adaptivity from ignoring important but low-variance directions
• Allows for a sparse coupling of p and u dependence
• Easily adaptable to more than two groupings of directions
• Different directional termination methods provide greater control of mixed terms
• Directional η values are also a useful diagnostic quantity

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