Cost of equity in homeland security resource allocation in the face of possible aon-strategic attacker

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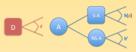
Introduction

- Research Objectives
  - Explore the guidelines of a centralized defender for optimal defensive resource allocation in the presence of an unknown hazard (strategic or non-strategic) [1] [3] **while reserving a portion for equal distribution, according to geographical areas (type 1), valuation (type 2), population (type 3), population density (type 4), and density-weighted population (type 5)**
  - Investigate applications of game theory to risk analysis for homeland security if the attacker is not rational or strategic [1]
- Game Theory - Basic Assumptions
  - **Both attacker and defender are fully rational?**

Game Formulation

- A sequential game, where the defender moves first by allocating defensive resources.
- A strategic attacker's best response



$$\hat h_i(d) = \begin{cases} \frac{1}{||P||} & \text{if } i \in P \equiv \{i : h_i > 0\} = \left\{i : p_i(d_i)x_i = \max_{j=1,2,\cdots,n}\{p_j(d_j)x_j\}\right\}, \\ 0 & \text{otherwise.}\end{cases}$$

where $||P||$ is the cardinality of set P.

- $p_i(d_i) = e^{-\lambda d_i}$: Success probability of an attack on target i, $i = 1,2,\cdots,n$.
- x_i: Valuation for target i, for $i = 1,2,\ldots,n$.
- c_i: Defense allocation from the non-reserved budget to target i.
- $\tilde c_i$: Defense allocation from the reserved budget to target i.
- $d_i = c_i + \tilde c_i$: Government's entire defense resource allocation to target i, for $i = 1,2,\ldots,n$.
- $c \equiv (c_1, c_2,\cdots,c_n); \tilde c \equiv (\tilde c_1,\tilde c_2,\cdots,\tilde c_n); d \equiv (d_1,d_2,\cdots,d_n)$.
- P: Set of targets that a strategic terrorist attack with strictly positive probabilities. That is,
$$P \equiv \{i : h_i > 0; i = 1,2,\ldots,n\} \qquad (1)$$
- I_i: Indicator function for the event $\{h_i \ge 0\}$, $i = 1,2,\ldots,n$. That is,
$$I_i = \begin{cases} 1 & \text{if } h_i > 0 \\ 0 & \text{if } h_i = 0\end{cases} \qquad (2)$$

- Strategic vs. Non-Strategic Attacker
  - Strategic attack probabilities are endogenously-determined by the game and denoted as $h_i(d)$ for target i.
  - Non-strategic (NS) attack probabilities are exogenously-given and denoted as h'_i for target i.

Model, Analytic Results, Numerical Illustrations & Robustness Analysis

The Defender's Optimization Problem

$$\min_d \; L(d,\hat h(d),e,q) = q\underbrace{\sum_{i=1}^n \hat h_i(d)p_i(d_i)x_i}_{\text{[strategic term]}} + (1-q)\underbrace{\sum_{i=1}^n h'_i p_i(d_i)x_i}_{\text{[non-strategic term]}}$$

subject to: $\sum_{i=1}^n d_i = D; d_i \ge \tilde c_i$

Urban area equity: $\tilde c_i = \frac{eC}{n}$ Per-valuation equity: $\tilde c_i = \frac{eCx_i}{\sum_{i=1}^n x_i}$

Per-capita equity: $\tilde c_i = \frac{eCp_i^s}{\sum_{i=1}^n p_i^s}$ Per-density equity: $\tilde c_i = \frac{eCp_i^d}{\sum_{i=1}^n p_i^d}$

Per-adjusted-capita equity: $\tilde c_i = \frac{eCp_i^w}{\sum_{i=1}^n p_i^w}$

- D: Total budget of the defensive resources.
- p_i^s: Population size of target i, for $i = 1,2,\ldots,n$.
- p_i^d: Population-density of target i, for $i = 1,2,\ldots,n$.
- p_i^w: Density-weighted population size of target i, for $i = 1,2,\ldots,n$.

Analytic Results

Theorem 1 *Given $e \neq 1$, consider a pair of equilibrium strategies, (h^*, d^*), and the associated variables, R_i^*, A_i^* and J^*. We have $R_i^* \equiv A_i^* p_i(d_i^*)x_i = W^*$, $\forall i \in J^*$ Moreover, we have $R_i^* \le W^*$ for all $i \notin J^*$.*

A_i: Total attack probability, attack probability of a strategic terrorist, or normalized marginal risk reduction from an attack by a non-strategic terrorist for target i.

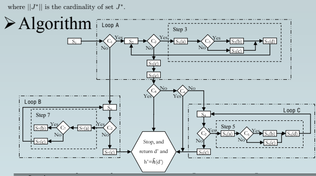
$$A_i \equiv \begin{cases} qh_i + (1-q)h'_i & \text{if } q \in (0,1], \text{ and } J \neq P \\ qh_i & \text{if } q \in (0,1], \text{ and } J = P \\ \frac{dp_i(c_i)}{dc_i}\frac{h'_i}{p_i(c_i)} & \text{if } q = 0\end{cases}$$

J: Set of defended targets with non-reserved defensive resources. That is,
$$J \equiv \{i : c_i > 0; i = 1,2,\ldots,n\}$$

$$W^* = \exp\left\{\frac{\sum_{i\in J^*}\ln x_i + \sum_{i\in J^*}\ln h_i^*(d) - \lambda(1-e)D - \sum_{i\in J^*}\tilde c_i}{||J^*||}\right\} \qquad (3)$$

where $||J^*||$ is the cardinality of set J^*.

Algorithm

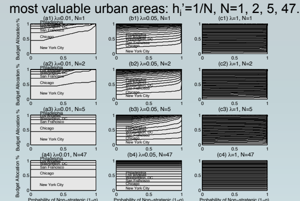


Steps	S_1	Let $J \equiv \{1,2,\ldots,n\}$, $P \equiv \{1,2\ldots,n\}$ and $I \equiv \{1,1,\ldots,1\}$.
	S_2	Solve for W using Equation (3) with $A_i = q\frac{I_i}{\|\|P\|\|} + (1-q)h'_i$.
	S_3	(a) Identify such index j('s) satisfying C_2 with smallest x_j. Let $c_j = 0$. Delete j('s) from set J.
		(b) Let $c_j = 0$ and delete j('s) from set J.
		(c) Save old c_j and update $c_j = \frac{1}{\lambda_j}\left\{\ln x_j + \ln\left[q\frac{I_j}{\|\|P\|\|} + (1-q)h'_j\right] - \ln W\right\}$, $\forall j \in J$.
		(d) Update sets P and I using Equations (1-2) excluding most valuable targets attracting NS-terrorist.
		(e) Replace $c_j \forall j \in J$ with old c_j's and adjust c_j's so that $p_j(c_j)x_j = \max\{x_i\} \forall j \in J - P, i \notin J$.
	S_4	(a) Let $J \equiv \{1,2,\ldots,n\}$, $P \equiv \{1,2\ldots,n\}$ and $I \equiv \{1,1,\ldots,1\}$.
		(b) Solve for W using Equation (3) with $A_i = q\frac{I_i}{\|\|P\|\|}$.
	S_5	(a) Identify such index j('s) satisfying C_2 with smallest x_j. Let $c_j = 0$. Delete j('s) from set J.
		(b) Let $c_j = 0$ and delete j('s) from set J.
		(c) Update $c_j = \frac{1}{\lambda_j}\left[\ln x_j + \ln\left(q\frac{I_j}{\|\|P\|\|}\right) - \ln W\right]$, $\forall j \in J$.
		(d) Update sets P and I using Equations (1-2).
	S_6	Solve for W using Equation (3) with $A_i = \lambda h'_i$.
	S_7	(a) Identify such index j('s) satisfying C_2 with smallest x_j. Let $c_j = 0$. Delete j('s) from set J.
		(b) Let $c_j = 0$ and delete j('s) from set J.
		(c) Update $c_j = \frac{1}{\lambda_j}(\ln x_j + \ln \lambda h'_j - \ln W)$, $\forall j \in J$.
Conditions	C_1	$q \in (0,1]$.
	C_2	There exists some index $j \in J$ such that $x_jA_j \le W$.
	C_3	$q\frac{I_j}{\|\|P\|\|} + (1-q)h'_j = 0 \; \forall j \in J$.
	C_4	$J \neq P$ and $q\frac{I_j}{\|\|P\|\|} < (1-q)h'_j \; \forall i \in P$ and $j \in J - P$.
	C_5	$I_j = 0, \forall j \in J$.
	C_6	$J = P$ and $L(c,h(c),h',q) \le L^*(c^*,h(c^*),h',q=1)$.
	C_7	$h'_j = 0 \; \forall j \in J$.
	C_8	$x_i > p_j(c_j)x_j \; \forall i \notin J, j \in J$

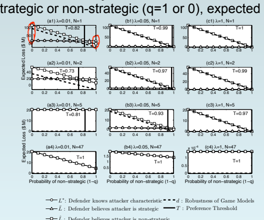
Data Sources: Expected property loss for 47 U.S. urban areas [2] and FY2004 UASI Grant Allocation [4]

#	Urban Areas	$x_i^\dagger$	Population†	$p_i^{d\dagger}$	Density-Weighted Population†	FY 2004 UASI Allocations ($)‡
1	New York City	413.0	9,314,235	8,159	75,991,762,554	47,007,064
2	Chicago	115.0	8,272,768	1,634	13,519,096,414	34,142,222
3	San Francisco	57.0	1,731,183	1,705	2,951,064,038	26,481,275
4	Washington, D.C.	36.0	4,923,153	756	3,723,526,125	29,301,502
5	Los Angeles-Long Beach	34.0	9,519,338	2,344	22,314,867,674	40,404,595
6	Philadelphia, PA-NJ	21.0	5,100,931	1,323	6,749,136,215	23,078,759
7	Boston, MA-NH	18.0	3,406,829	1,685	5,740,709,241	19,131,723
8	Houston	11.0	4,177,646	706	2,948,039,040	19,955,485
9	Newark	7.3	2,032,989	1,289	2,619,713,383	15,054,101
10	Seattle-Bellevue	6.7	2,414,616	546	1,318,032,823	16,516,007
11	Jersey City	4.4	608,975	13,044	7,943,237,618	17,112,311
12	Detroit	4.2	4,441,551	1,140	5,062,484,593	13,754,597
13	Las Vegas	4.1	1,563,282	40	62,076,079	10,531,025
14	Oakland	4.0	2,392,557	1,642	3,927,449,645	7,854,691
15	Orange County	3.7	2,846,289	3,606	10,262,626,470	25,404,219
16	Cleveland	3.0	2,250,871	832	1,871,707,337	10,460,465
17	San Diego	2.8	2,813,833	670	1,885,205,299	10,479,947
18	Miami	2.7	2,253,362	1,158	2,609,185,020	20,108,247
19	Minneapolis-St. Paul	2.7	2,968,806	490	1,453,687,745	19,146,642
⋮		⋮	⋮	⋮	⋮	⋮
47	Fresno	0.2	922,516	114	105,084,482	7,076,396
	Total	788.7	122,581,611	58,281	200,768,260,341	**675,000,000**

Illustrations (e=0): NS attack probability: top N most valuable urban areas: h_i'=1/N, N=1, 2, 5, 47.

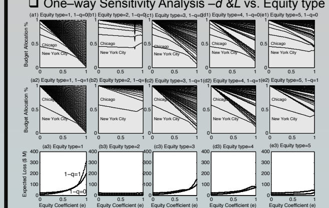


Robustness Analysis: Falsely believes attacker is strategic or non-strategic (q=1 or 0), expected loss:

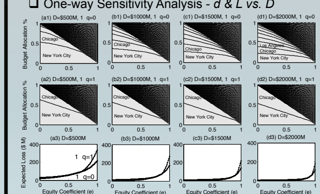


Illustrations (0≤e≤1):

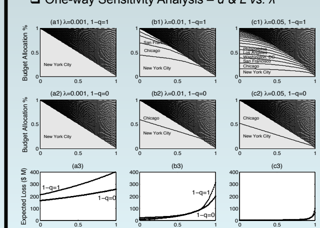
- One–way Sensitivity Analysis –*d &L* vs. Equity type



- One-way Sensitivity Analysis - *d & L vs. D*



- One-way Sensitivity Analysis – *d & L vs. λ*



Summary & Conclusions

- Assess the robustness of game theory when its assumption of rationality is violated
  - Game models are preferred even when the probability of a non-strategic attacker is greater than 50%
  - Defensive resource allocation based on game theory will not incur much additional expected loss in a large range of parameter values as compared to non-game theory
- Defender trades off between efficiency and equity
  - Per-target valuation equity (type 2) is preferred over other types of equity
  - Equity is costly and such cost increases convexly in equity
  - When the defense is not effective, cost of equity is very high

References

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[4] Office of Grant and Training. 2004. Overview: FY 2004 homeland security grant program.

Acknowledgments

This research was funded through the National Center for Risk and Economic Analysis of Terrorism Events (CREATE) Center of Excellence by a grant from the Department of Homeland Security, Science and Technology Directorate, Office of University Programs under grant number 2007-ST-061-000001. However, any opinions, findings, and conclusions or recommendations in this document are those of the authors and do not necessarily reflect views of the United States Department of Homeland Security. We thank Prof. Vicki, M. Bier (University of Wisconsin-Madison) for helpful comments.

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