Constrained Optimization and Bayesian Updating for Conditioning and Learning

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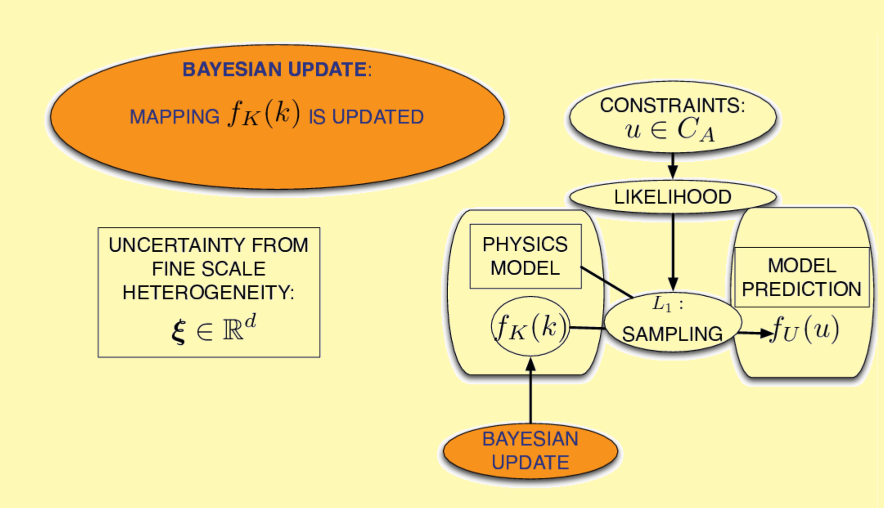
Objectives & Challenges

Polynomial Chaos Representations (PCE) are a key tool at the intersection of UQ and CSE. Computing PCE representations, while often computer intensive, yields representations of stochastic processes and random variables that are accurate representations of a functional dependence between random variables as well as carry a convergent approximation of target probability measure.

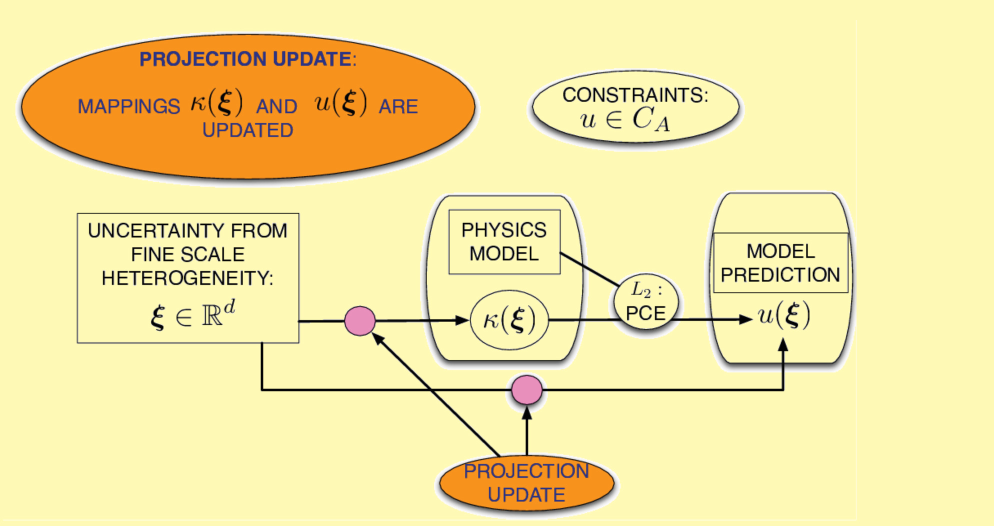
Traditional probabilistic updating scheme, including Bayesian methods, only require the probabilistic content of these approximation to form the prior and likelihood functions. The functional dependence between input and output, while already computed at great expense, is not leveraged.

Capitalizing on the PCE construction, updating can be construed as a constrained optimization problem in a Hilbert space already described and constructed as part of the PCE formalism. We demonstrate this projective updating for PCE representations of both output predictions and model parameters.

Bayes and PCE Updates

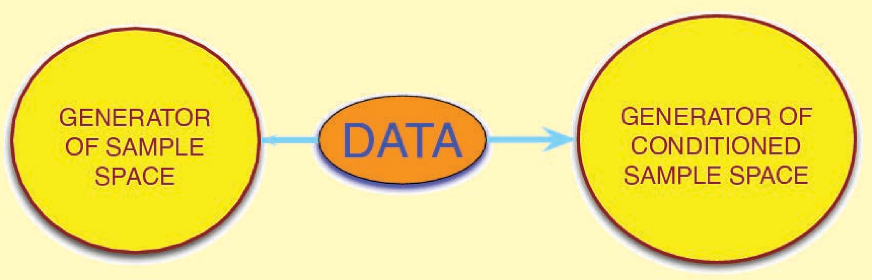


BAYESIAN UPDATE

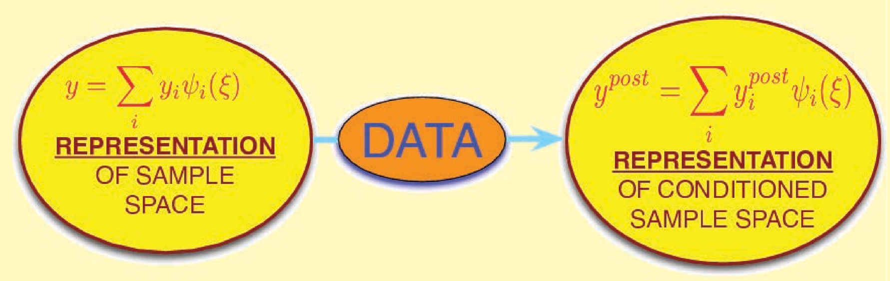


PCE UPDATE

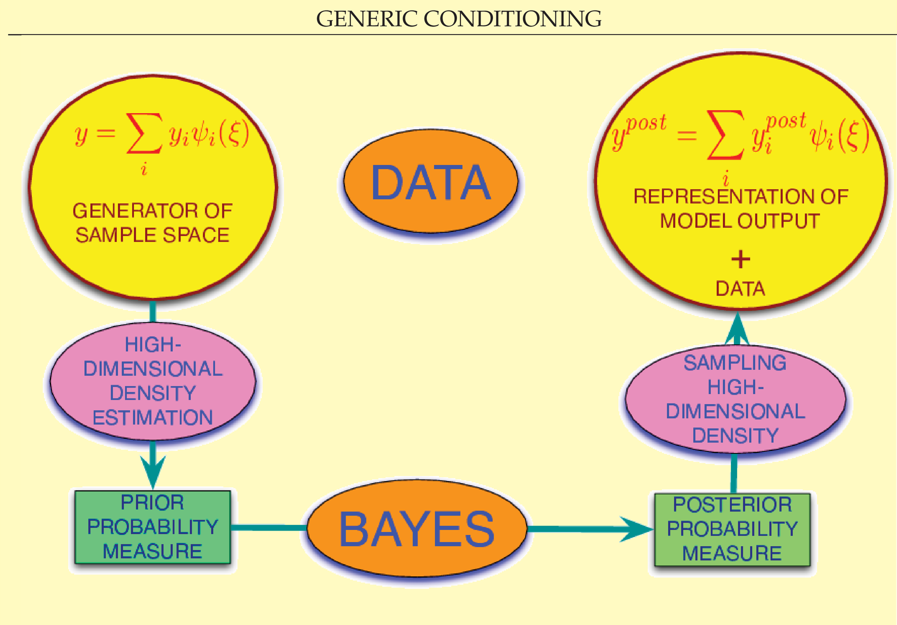
Updating Polynomial Chaos Representations



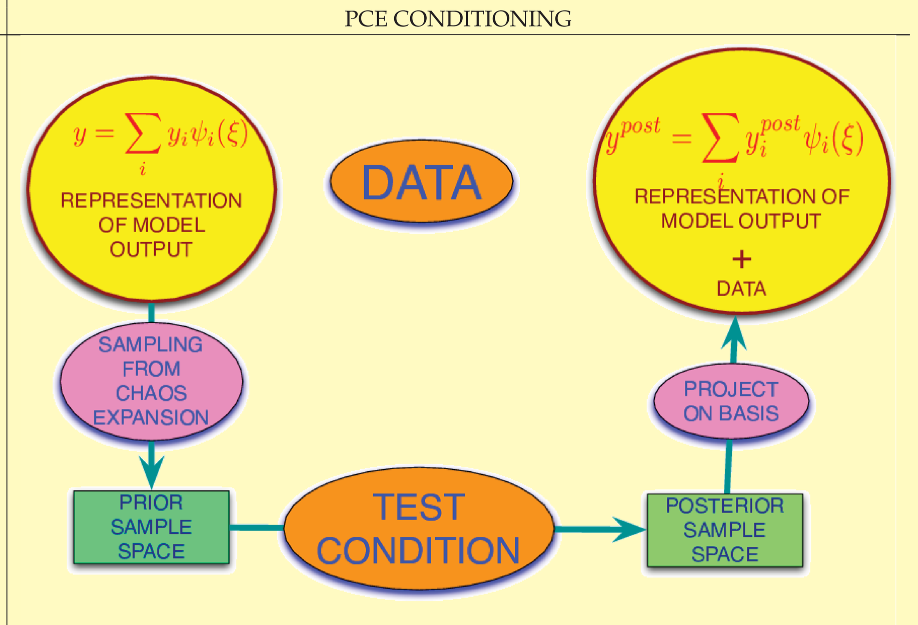
GENERIC CONDITIONING



PCE CONDITIONING



BAYESIAN UPDATE OF PCE



PROJECTION UPDATE OF PCE

Constrained Projection

PRIOR MODEL:

$$Y = \sum_{i=0}^{n} Y_i \psi_i(\boldsymbol{\xi}) \qquad \in \mathcal{P}_n(\mathbb{H})$$

CONSTRAINT:

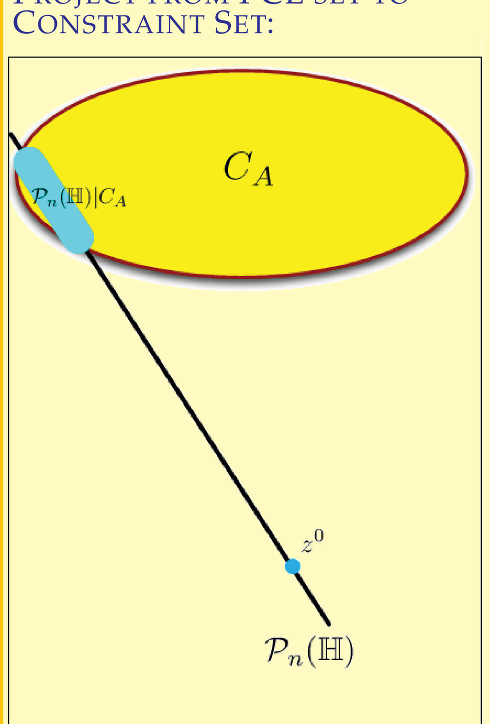
$$C_A = \{Y : Y \in A \ \text{ a.s.}\}$$

POSTERIOR MODEL:

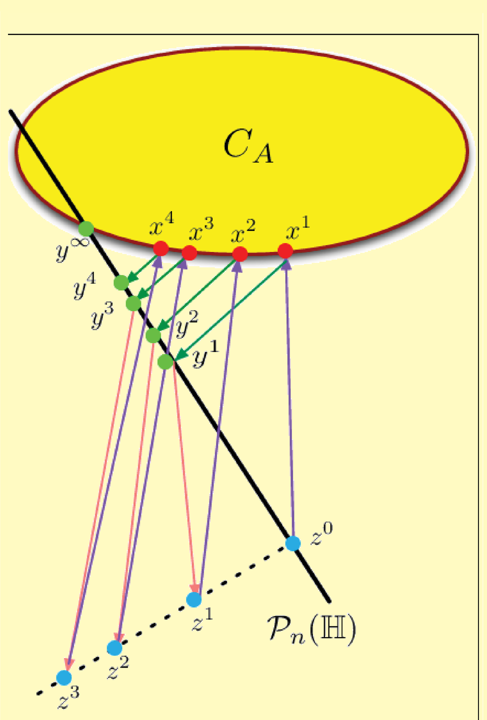
$$Y^{\text{post}} = \sum_{i=0}^{n} Y_i^{\text{post}} \psi_i(\boldsymbol{\xi}) \qquad \in \mathcal{P}_n(\mathbb{H})$$

Constrained Projection

Project from PCE set to constraint set:



Dykstra's Algorithm



Dykstra's Algorithm

DYKSTRA'S ALGORITHM:

$$\lim_{k\to\infty} \| Y^k - \mathbf{Proj}_{\mathcal{P}_n(\mathbb{H}) \cap C_A}(Y) \| = 0$$

INITIALIZE:

$$Z^0 = Y$$

PROJECT ON C_A:

$$X^{k+1} = \mathbf{Proj}_{C_A}(Z^k)$$

PROJECT ON $\mathcal{P}_n(\mathbb{H})$:

$$Y^{k+1} = \mathbf{Proj}_{\mathcal{P}_n(\mathbb{H})}(X^{k+1})$$

CORRECTOR:

$$Z^{k+1} = Z^k - (X^{k+1} - Y^{k+1})$$

Dykstra for PCE

INITIALIZE:

$$Z^0 = \sum_{i=0}^{p} y_i^0 \psi_i(\boldsymbol{\xi})$$

PROJECT ON C_A:

$$X^{k+1} = \sum_{i=0}^{m} x_i^{k+1} \psi_i(\boldsymbol{\xi}) = \mathbf{Proj}_{C^A}\left(\sum_{i=0}^{m} z_i^k \psi_i(\boldsymbol{\xi}) \right)$$

$$x_i^{k+1} = \int_\Omega \mathbf{Proj}_{C^A}\left(\sum_{j=0}^{m} z_j^k \psi_j(\boldsymbol{\xi}) \right) \psi_i(\boldsymbol{\xi}) d\boldsymbol{\xi}\,, i = 0, \cdots, m$$

PROJECT ON $\mathcal{P}_n(\mathbb{H})$:

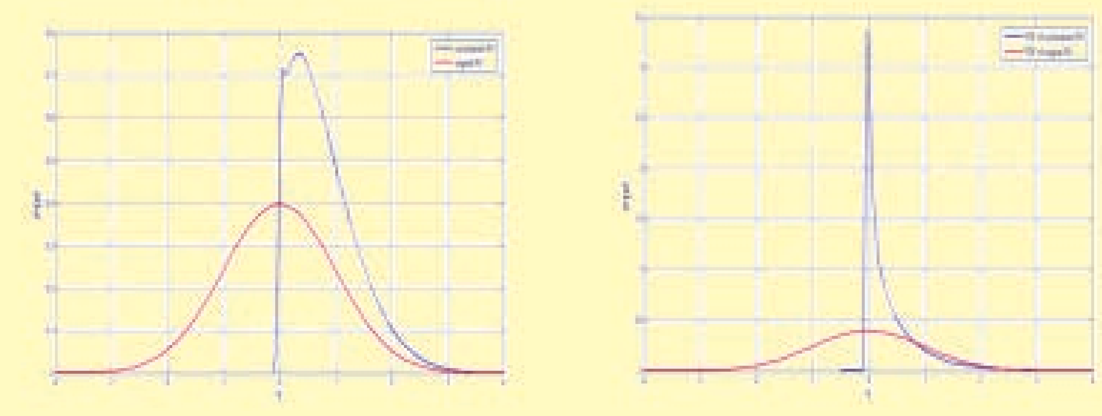
$$Y^{k+1} = \mathbf{Proj}_{\mathcal{P}_n(\mathbb{H})}(X^{k+1}) = \sum_{i=0}^{n} x_i^{k+1} \psi_i(\boldsymbol{\xi})$$

CORRECTOR:

$$Z^{k+1} = \sum_{i=0}^{m} z_i^k \psi_i(\boldsymbol{\xi}) - \sum_{i=n+1}^{m} x_i^{k+1} \psi_i(\boldsymbol{\xi}) = \begin{cases} z_i^k & \text{if } i \le n \\ z_i^k - x_i^{k+1} & \text{if } i > n \end{cases}$$

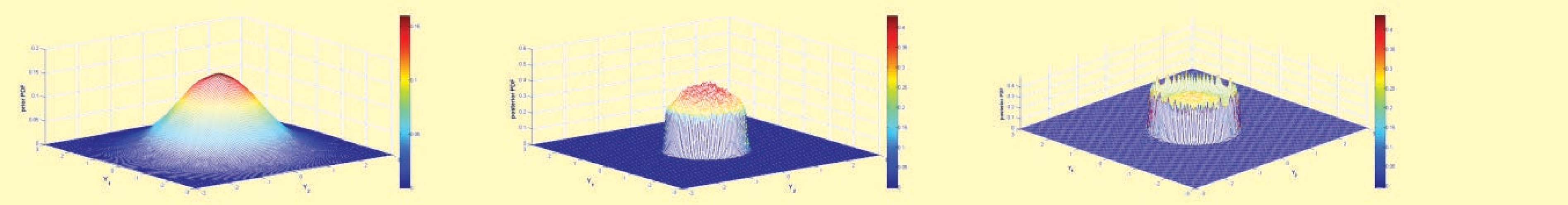
Examples

1D Gaussian prior with positive constraints:



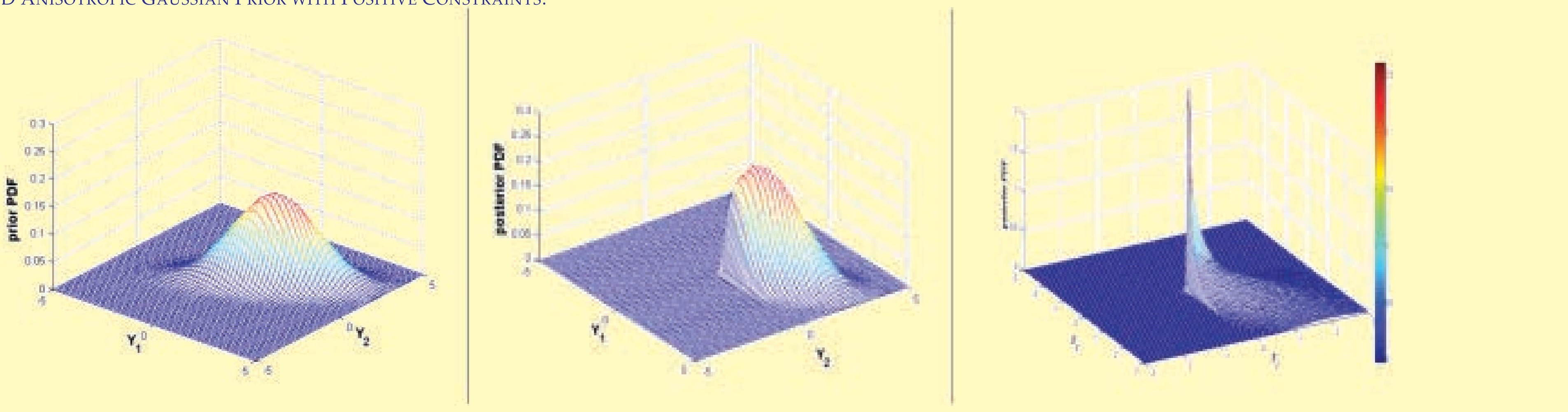
Bayesian Update Projected Update

2D Gaussian prior with ellipsoidal constraints:



Gaussian Prior Bayesian Update Projected Update

2D Anisotropic Gaussian prior with positive constraints:



Gaussian Prior Bayesian Update Projected Update

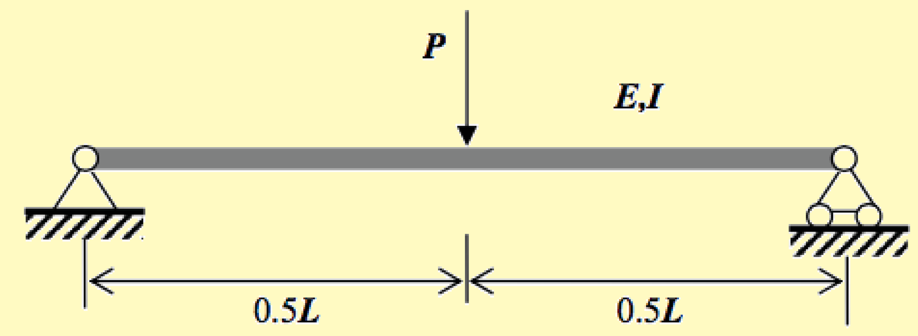
Update of Model Parameters

Consider a beam with random elastic coefficients. The maximum deflection at the center will be a random variable. We adopt a simple Euler-Bernoulli model for the beam, which only account for flexural effects. We assume a prior probabilistic model for elasticity modulus, and propagate that into a PCE model for the deflection of the beam.

The real beam is more complex. We know its center deflection is bounded between two experimental values. We update the prior probabilistic models with information.

We update both:

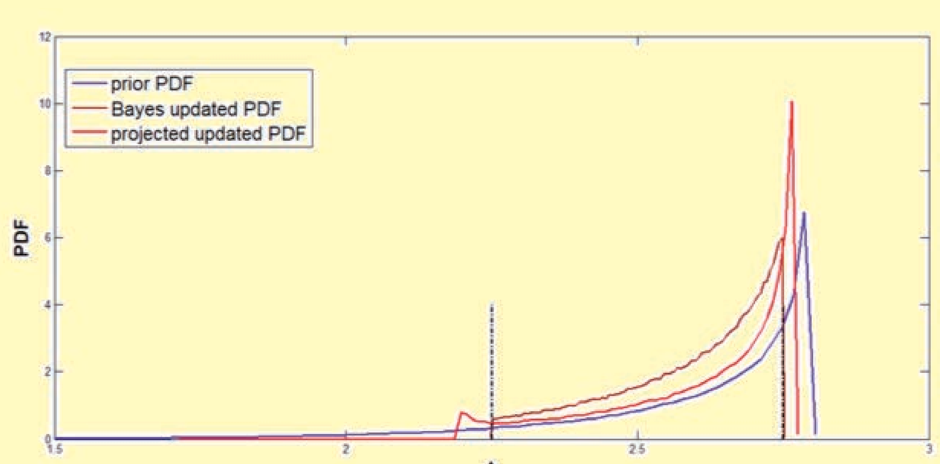
- predicted stochastic model of the deflection.
- prior stochastic model of elasticity parameter.



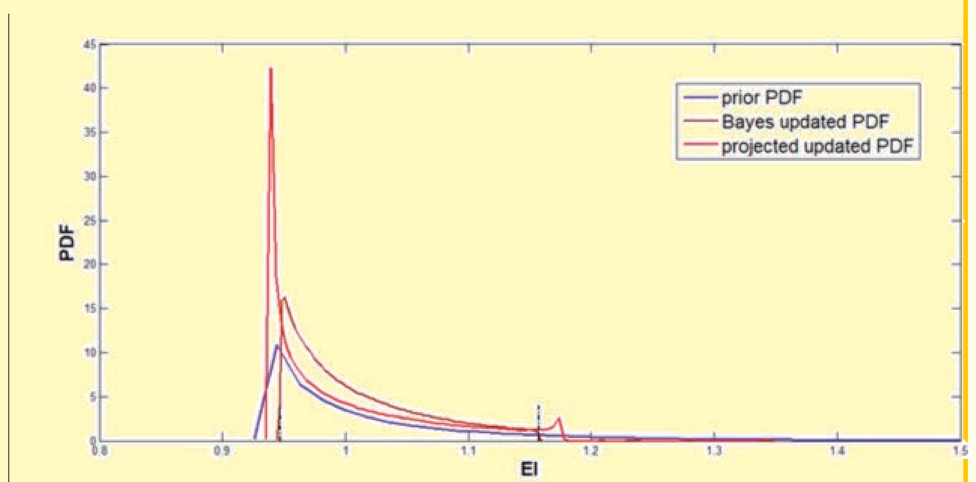
- PRIOR PROBABILISTIC MODEL:

$$E = \left(1 - \frac{\delta}{\sqrt{2}}\right)\bar{E} + \frac{\delta \bar{E}}{\sqrt{2}}\xi^2\,, \qquad \begin{cases} \xi \sim \mathcal{N}(0,1) \\ \delta = \sigma_E / \bar{E} \end{cases}$$

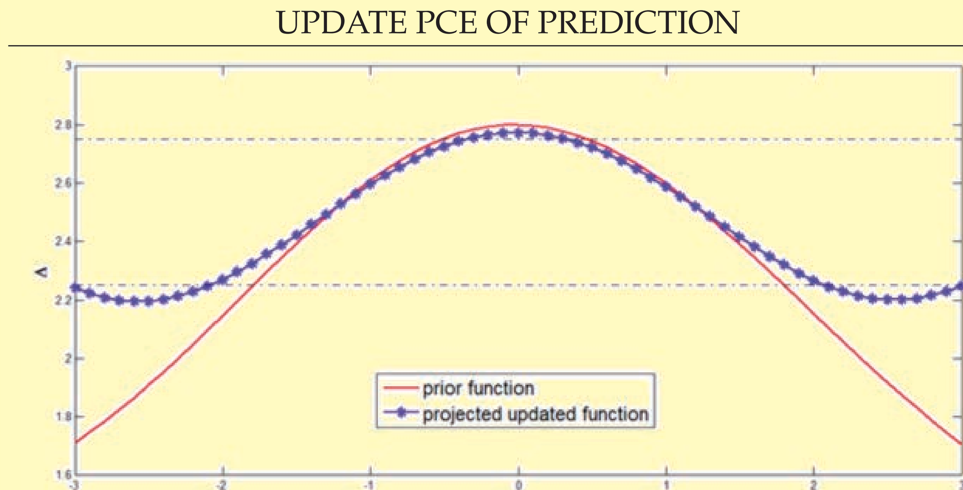
- CONSTRAINT:
  BOUNDS ON MAXIMUM DEFLECTION AT CENTERPOINT.



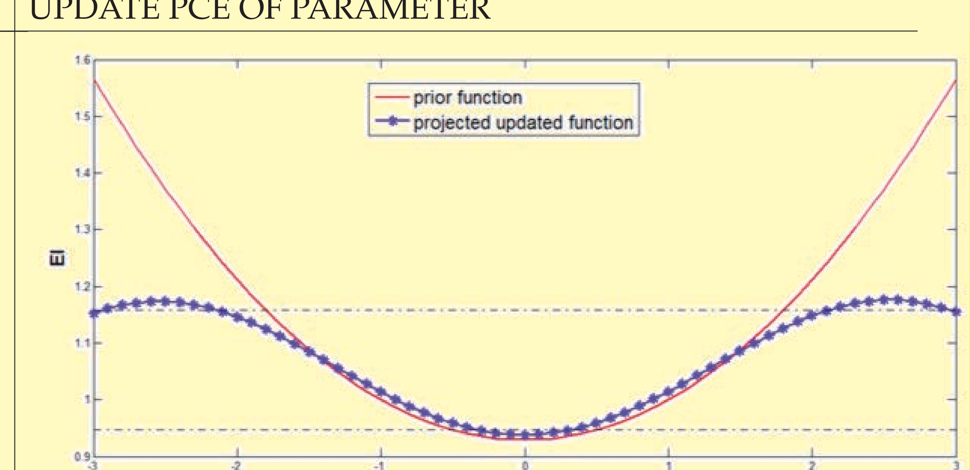
UPDATE PCE OF PREDICTION



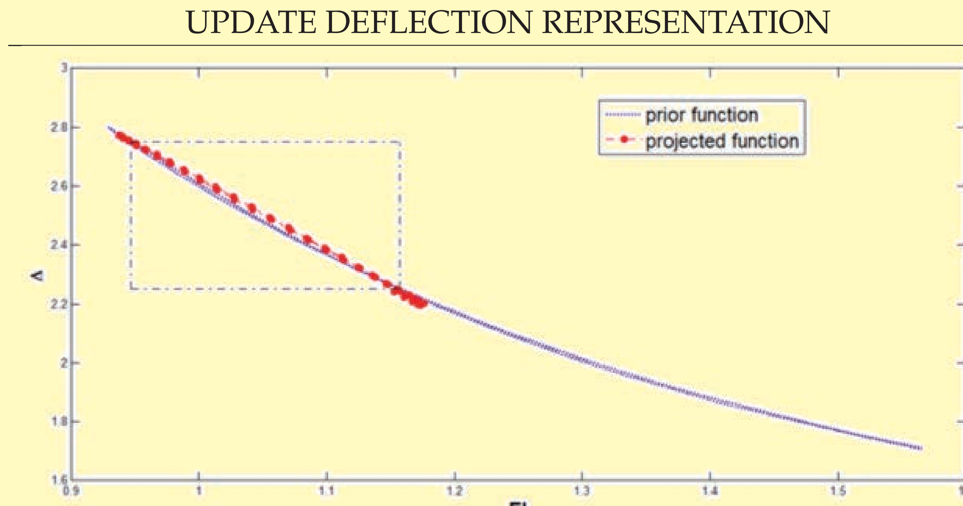
UPDATE PCE OF PARAMETER



UPDATE DEFLECTION REPRESENTATION



UPDATE MODULUS REPRESENTATION



UPDATED PHYSICS MODEL

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