Development and A Priori Testing of Sparse Adaptive Pseudospectral Polynomial Chaos Expansions

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Abstract

A crucial aspect in the application of uncertainty quantification methods to complex models concerns the ability to sample quantities of interest efficiently. This poster presentation discusses the development and implementation of a sparse, adaptive sampling strategy to efficiently and reliably construct suitable polynomial chaos surrogates. The approach relies on a pseudospectral construction that accommodates arbitrary admissible sparse grids. An a priori analysis of the implementation and performance of the algorithm is conducted, based on existing databases of ocean global circulation simulations. The tests demonstrate that in the present setting order-of-magnitude savings are obtained over isotropic sparse sampling.

Model Test Case: Hybrid Coordinate Ocean Model

Four Uncertain Parameters

	Parameter	Distribution
1	critical Richardson number	$\mathcal{U}(0.25, 0.7)$
2	background viscosity (m^2/s)	$\mathcal{U}(10^{-4}, 10^{-3})$
3	background diffusivity (m^2/s)	$\mathcal{U}(10^{-5}, 10^{-4})$
4	stochastic wind drag coefficient	$\mathcal{U}(0.2, 1.0)$

Gulf of Mexico during Hurricane Ivan (September 2004)

Polynomial Chaos Expansions

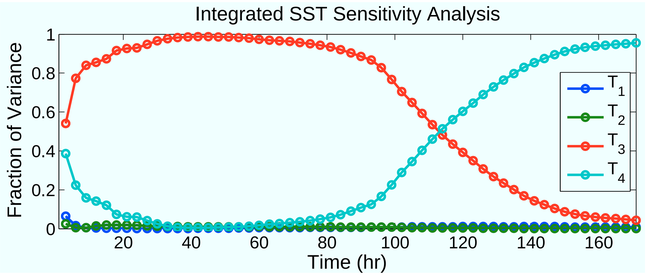
Fourier-like expansion over orthogonal basis function Ψ_k

$$U(\xi, t) = \sum_{k=0}^{P} U_k(\xi, t)\Psi_k(\xi)$$

Coefficients U_k given by $U_k = \frac{\langle U, \Psi_k \rangle}{\langle \Psi_k^2 \rangle} \approx \frac{\langle U, \Psi_k \rangle_Q}{\langle \Psi_k^2 \rangle}$ where $\langle \cdot, \cdot \rangle_Q$ represents scalar product performed with numerical quadrature.

Full Database Solution: Sensitivity Analysis

Database of 385 realizations for up to fifth order polynomials, using Smolyak quadrature, and equal resolution in all dimensions. T_i is the total sensitivity index due to the ith random variable.



- 3rd and 4th dimensions dominate sensitivity
- Can accuracy be reduced in first two dimensions?
- Should accuracy be increased in the latter two dimensions?

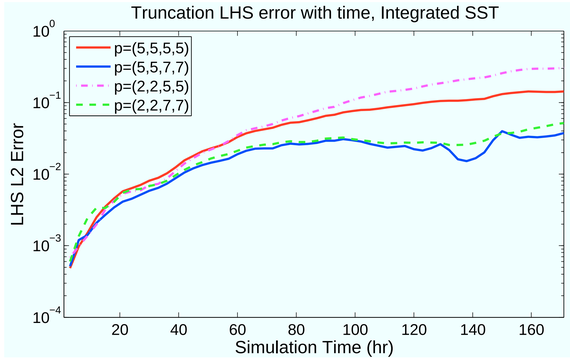
Dimensionally Adaptive Representation

- Enrich along 3rd and 4th dimensions (128 additional realizations)
- Generate Latin-Hypercube sample with 256 realizations for verification

Accuracy	P	N	Description
$(5,5,5,5)$	126	385	Original
$(5,5,7,7)$	168	513	Enriched in 3 & 4
$(2,2,5,5)$	36	73	Reduced in 1 & 2
$(2,2,7,7)$	59	169	Enriched and Reduced

P: Number of Polynomials
N: Number of Realizations

Dimensionally Adaptive Results

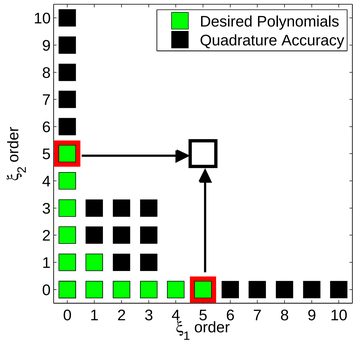


- Increased resolution along 3rd and 4th dimension yields lower error
- Decreased resolution along 1st and 2nd dimension has minimal affect on error

General Adaptivity

The dimensionally adaptive approach may still include more realizations than necessary. However, one must be careful in order to minimize aliasing effects

- Internal aliasing can only be avoided if scalar products of **all** polynomial pairs can be correctly integrated
- Aliasing errors are same order of magnitude as coefficients
- Note: There is no internal aliasing incurred when performing a "rectangular" projection.



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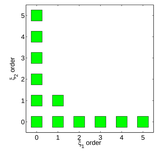
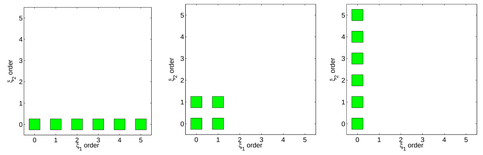
Smolyak Pseudospectral Projection

Instead of a direct Smolyak quadrature, build final projection as a Smolyak sum of internal-aliasing-free projections

Example:



is a weighted superposition of



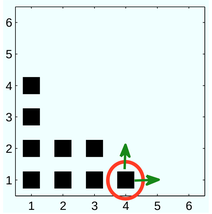
and all smaller projections

Remarks:

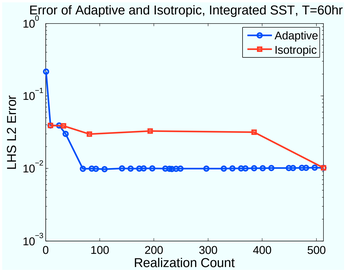
- In addition to avoiding internal aliasing, the Smolyak pseudospectral projection permits higher-order polynomials (especially monomial terms) with *the same realization stencil*
- Inherent in construction is an error indicator for adaptive refinement

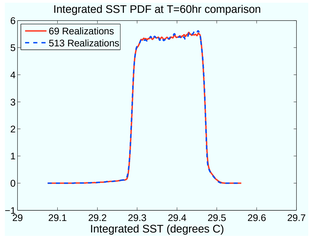
Adaptive Refinement

1. Start with a base representation
2. Choose multi-index with highest indicator
3. Enrich index with forward neighbors
   - Require the index remains admissible for proper telescoping
   - Require associated realizations be in database (for a priori testing)



Adaptive Driver Results





Conclusions:

- Reach error threshold with 69 realizations compared to 513
- Probability Density Function estimations are nearly identical
- Adaptive refinements leads to order-of-magnitude savings with **no prior knowledge** of full solution

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